Social Choice in Combinatorial Domains

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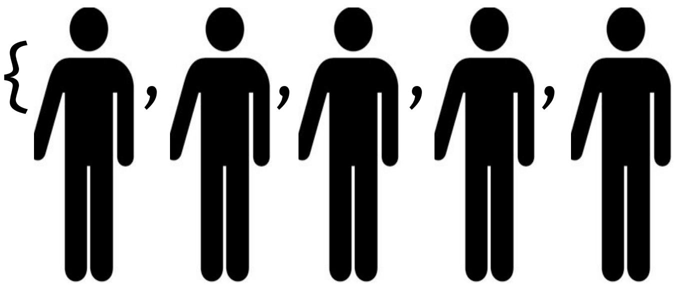
21.11.2012

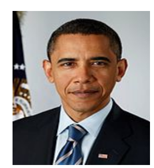
Outline:

- Introduction
- Paradoxical outcomes
- Compact way of preference representation
- Approach in Multi-winner elections
- Summary

Introduction

We have seen:

A set of individuals: N={  …}

A set of alternatives: X={  ,  , … }

And the winner was Obama

Voters vote only for one alternative!!!

Introduction

- How about asking voters to give their preferences for two or more alternatives!!!
- For example, Referendum:
  - # of proposals (new road, church, school …)
  - Rejected vs. Approved

Introduction

Let's have a look to a referendum example:

- A set of voters/individuals
- A number of proposals …(m)
- Approved vs. Rejected
- We get : 2^m

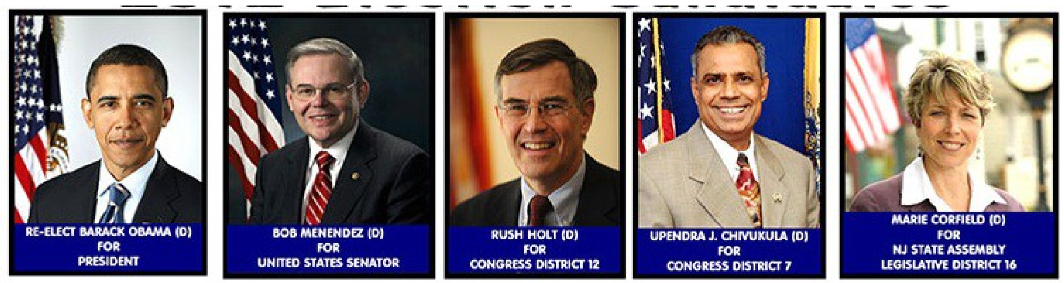
  Possible outcomes

Introduction

Even more …

- If there **m** seats in a committee

- **n** candidates



- Number of possible outcomes?: m^n
- The number of actual alternatives to consider is at least exponential in **m**

Introduction

We need :

- Compact way of preference representation
- Methods of defining the outcomes of Multi-Winner elections
  - More effective than explicit way

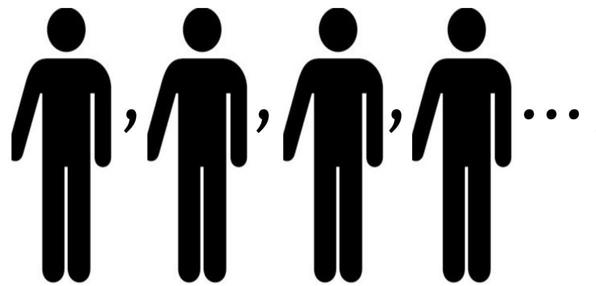
Definitions

For ease of exposition:

- Consider binary combinatorial domains
  - Each issue takes value=: 1 (“yes”) or 0 (“no”)
  - Let L be a finite set of such binary issues
  - Each issue $k \in L$ associated with a variable
  - X_k ={1 , 0}
    - x_k ***means*** X_k ***=1***
    - x'_k ***means*** X_k ***=0***

Example: Multi-Winners

- **Voters:**



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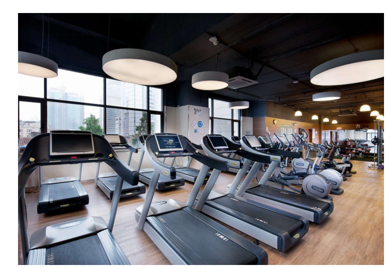
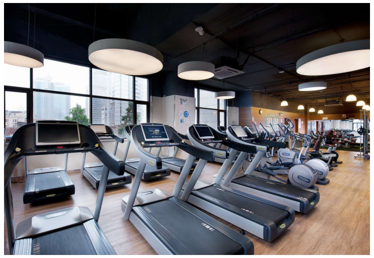
- **Binary Issues:**





Example: Multi-Winners

Results:

- 1.  2.  3. 

(1,0,0), (0,1,0), (0,0,1)	*3 votes*
(1,1,1), (1,1,0), (1,0,1), and (0,1,1)	*1 vote*
(0,0,0)	*0 vote*

Example: Multi-Winners

Using simple majority rule for each issue we get:

1. 

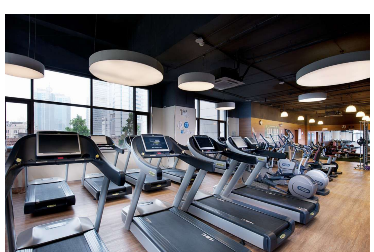
7 votes "no": $(0,0,1)_3 + (0,1,0)_3 + (0,1,1)_1$

6 "yes": $(1,0,0)_3 + (1,1,0)_1 + (1,0,1)_1 + (1,1,1)_1$

2. 

7 votes "no":

6 votes "yes":

3. 

7 votes "no":

6 votes "yes" (0,0,0) - winner

Paradox

- Paradox of Multiple Elections
- Why the above example is a Paradox
  - Voter's happiness is proportional to the number of issues her choice coincides
  - (0,0,0) could be reasonable results
- What is Paradox?

Definition

Grandi and Endriss: what is a Paradox?

Integrity Constraint: $X_1 \vee X_2 \vee X_3$

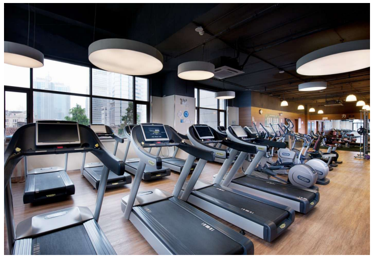
Individual ballots: (0,0,1) , (0,1,0) … (1,1,1)

Paradox:

- All individual ballots satisfy given integrity constraints
- But the outcome of election (0,0,0) does not

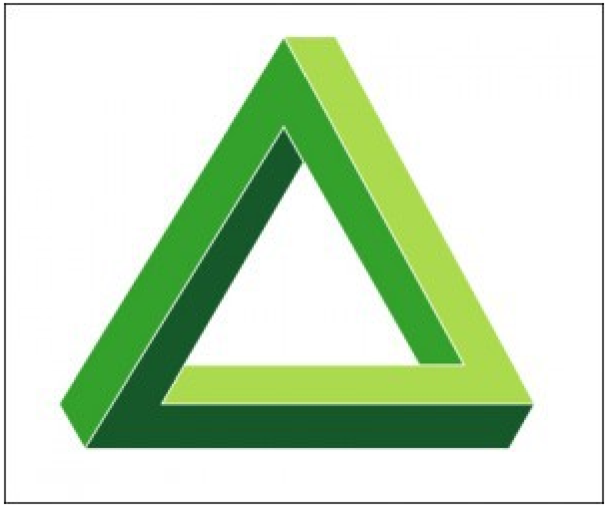
Example: Multi-Winners

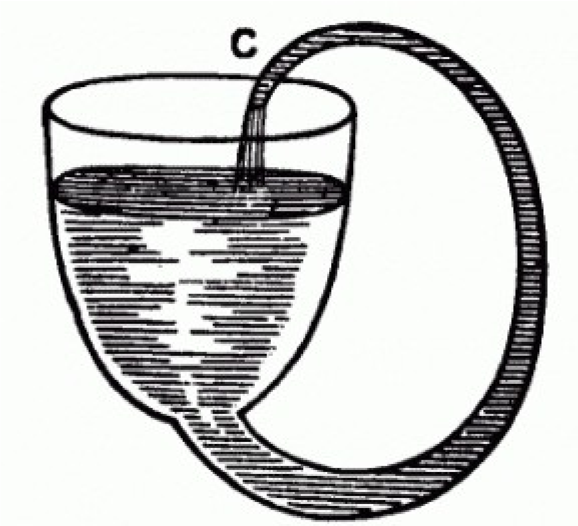
Results:

- 1.  2.  3. 

(1,0,0), (0,1,0), (0,0,1)	*3 votes*
(1,1,1), (1,1,0), (1,0,1), and (0,1,1)	*1 vote*
(0,0,0)	*0 vote*

- The election outcome: (0,0,0)





How to avoid this Paradox???

Avoid The Paradox

An approach:

- Directly vote on combinatorial alternatives:
  - Apply plurality rule to **Example 1:**
  - (1,0,0) , (0,1,0) and (0,0,1) are winners by 3 points:
  - Pick a winner by tie-breaking rules
  - Whatever we pick , the paradox is avoided.

Problem

- Not good approach in practice/general
  - It relies excessively on tie-breaking rule
  - 10 issues, 20 individuals
  - Assuming every profile is equally likely to occur
  - Probability that every combinatorial alternatives goes to tie-breaking is very high

Avoid The Problem

- Use other methods rather than plurality rule
- Methods that extract more information from the voters
  - Voters provide complete ranking of all alternatives
  - Recall: Borda count (n)-1 points for first rank,
  - (n)-2 Points for second rank and so on …
  - But for 10 issues, each voter must rank a total **1024** combinatorial alternatives, which is not realistic requirement.

Modeling preferences

- *We have seen several methods to avoid paradoxical outcomes*
- *Also we have seen how problematic they can be*
- *The fact is simply getting individual's preferences is challenging*
- *Model preferences*

Languages for Compact Preference Representation

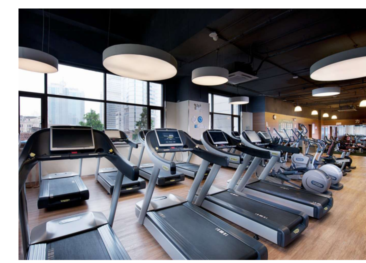
- Formal language used:
  - To express a given class of preference structures
  - Takes less space than explicit representation
- Wanted from voters:
  - Which combinatorial alternatives is acceptable, which is not (dichotomous preference structure)
  - 2 power of 2^m structure
    - Hence, takes 2^m bits to encode structures
- Is one of the major research areas of AI

Compact Preference Representation Language

- Conditional Preference Networks – {CP-nets}
  - The most widely used
  - Consists of Directed Graph
    - Where nodes are issues (X, Y, Z)
    - Each issue has a table of preferences (x > x')

CP-nets: Example

- Three binary issues X,Y,Z:
- Recall: x ={X=1} and x' ={X=0}



$$X \rightarrow Y \rightarrow Z, \quad X \rightarrow Z$$

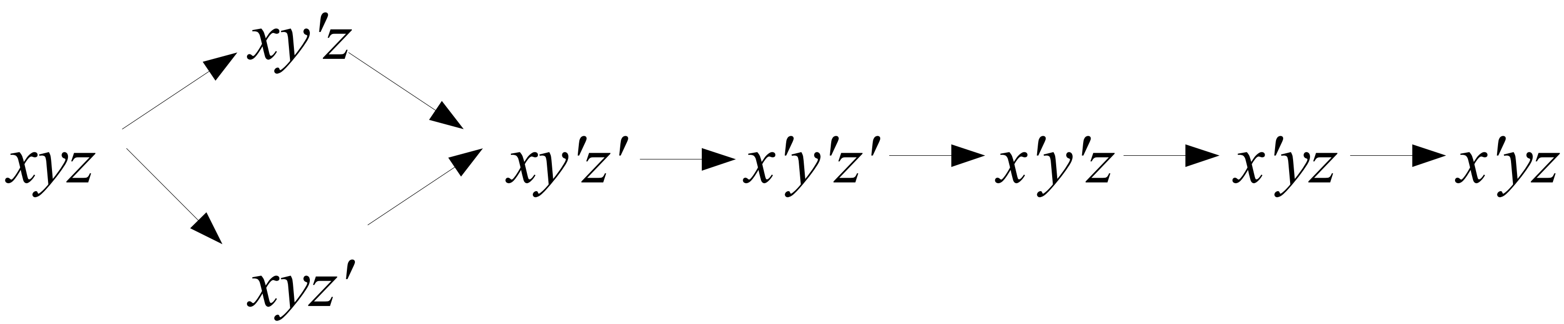
X	Y	Z
$x > x'$	$x : y > y'$	$xy : z > z'$
	$x' : y' > y$	$xy' : z > z'$
		$xy : z > z'$
		$x'y' : z' > z$

CP-tables

CP-nets generate:

Partial Order:

$\longrightarrow$ representing >

- 

For example xyz' > xy'z' equivalent for saying

(1,1,0) is preferred to (1,0,0)

- **Note:** We can not rank xy'z and xyz'

CP-nets:

- Express a large class of partial orders
  - Even though not all of them

- If the number of variables on which any single variable may depend on, is relatively small
  - Representation in of CP-nets will be relatively compact

Other approaches:

- *Using propositional language over variables to express goals*
  - *Goal:*
    - $X_1 \vee X_2$: *a goal to accept at least one of the two issues*
  - *Assign weights/priority to each goal*
  - *Weights/priority - we can represent a wide range of preferences*

Weights/Priority:

- Set of weighted goals
  - Utility of Combinatorial alternative is the sum of the weights of goals that are satisfied by that alternative
  - The combinatorial alternative with most sum is picked
- Set of goals labeled with priority levels
  - A combinatorial alternative X is preferred to Y
  - If there exists a level **L** that for each level of higher priority both satisfy the same number of goals
  - But in level **L** X satisfies more goals

Map:

- Where we are:
  - Compact preference representation

- Where to go:
  - How to use them
  - Find the outcome of Multi-Winner elections

Possible Approaches to Social Choice in Combinatorial Domains

Combinatorial vote:

- The idea:
  - To ask all individuals to express their preferences
    - In terms of a given compact preference representation language
  - To apply a voting rule of choice to those representations

Borda Rule and Prioritized Goals

- Three voters are asked to decide on two binary issues {X and Y}
- The voters express their preferences by giving prioritized goals
  - **1** indicates high priority, and **0** – normal priority
- Voters preferences:

$$Voter\ 1: \{X : 1,\ Y : 0\}$$

$$Voter\ 2: \{X \vee \neg Y : 0\}$$

$$Voter\ 3: \{\neg X : 0,\ Y : 0\}$$

Borda Rule and Prioritized Goals

Compact representation:

$$Voter\ 1: \{X : 1, Y : 0\}$$

$$Voter\ 2: \{X \vee \neg Y : 0\}$$

$$Voter\ 3: \{\neg X : 0, Y : 0\}$$

Weak orders (Lexicographic interpretation):

$$Voter\ 1: x\,y > x\,y' > x'\,y > x'\,y'$$

$$Voter\ 2: x\,y' \sim x\,y \sim x'\,y' > x'\,y$$

$$Voter\ 3: x'\,y > x'\,y' \sim x\,y > x\,y'$$

Borda Rule and Prioritized Goals

Applying Borda Rule:

- Can not be applied
  - Linear order
- Generalization of Borda Rule:
  - Alternative gets as many points from other alternatives it dominates

Results

- Weak order

$$Voter\ 1: x\,y > x\,y' > x'\,y > x'\,y'$$

$$Voter\ 2: x\,y' \sim x\,y \sim x'\,y' > x'\,y$$

$$Voter\ 3: x'\,y > x'\,y' \sim x\,y > x\,y'$$

Alternative	*Voter 1*	*Voter 2*	*Voter 3*	*Sum:*
$x\,y$	*3*	*1*	*1*	*5*
$x\,y'$	*2*	*1*	*0*	*3*
$x'\,y$	*1*	*0*	*3*	*4*
$x'\,y'$	*0*	*1*	*1*	*2*

Combinatorial Voting

- The above example is an illustration of combinatorial voting approach
  - Except transformation of compact into explicit representation
- Full implementation requires:
  - Algorithm for computing Borda winner directly from preferences that represented in prioritized goals
  - No such algorithm exists yet

Sequential Voting

- Is another important approach
  - The idea is:
    - Sequentially vote on each issue (local election)
    - Announce the decision on each issue
  - The simple rule to use for each “local election”
    - Two combinatorial alternatives
    - Rejected vs. Approved
    - The simple majority rule (May's Theorem)

Results on sequential voting

- *There is a lot of work and research have been done for sequential voting*
- *Here we briefly discuss two basic results*

Definitions

- A Condorcet Winner :
  - An alternative that would win against any other alternatives in majority contest
  - If it does exists, than we hope that it will be elected
  - (Recall: it does not need to necessarily exist)
- A Condorcet Loser:
  - Is an alternative that would lose against any other alternatives in majority contest
  - If it does exists, than we hope that it does not get elected

First Basic Result

- When:
  - All issues are binary
  - Simple Majority Rule is used in each local elections
    - local voting rule has a property of never electing a local Condorcet loser
  - Then, we will never elect a combinatorial alternative that is Condorcet Loser
- Example: Final local election - two combinatorial alternatives left
  - One of them is Condorcet loser, and by assumption on local rule, Condorcet Loser cannot win

Second Basic Result

- Voters express their preferences:
  - CP-nets
    - If the graph specifying them are all acyclic
    - If there exists an ordering of the issues that is compatible with all of them
  - Local Condorcet Winner by local voting rule is elected
  - Whenever there exists an alternative that is Global Condorcet Winner, the alternative will be elected

Summary

- Paradox of Multiple Elections
- Compact Preference representation
  - CP-nets
  - Goal based representation language
- Possible Approaches to Social Choice in Combinatorial Domains
  - Combinatorial Voting, (Borda Rule and Prioritized Goals)
  - Sequential Voting
    - Two Basic Results

Thanks for your attention!

Any questions?